Exponential energy decay of solutions for a system of viscoelastic wave equations of Kirchhoff type with strong damping

Gang Li, Linghui Hong, Wenjun Liu[1]

College of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China. E-mail: wjliu@nuist.edu.cn.

Abstract: The initial boundary value problem for a system of viscoelastic wave equations of Kirchhoff type with strong damping is considered. We prove that, under suitable assumptions on relaxation functions and certain initial data, the decay rate of the solutions energy is exponential.

Keywords: viscoelastic wave equation; Kirchhoff type; strong damping; exponential decay

AMS Subject Classification (2010): 35B40; 35L70; 35L53

1 Introduction

In this work, we investigate the following system of viscoelastic wave equations of Kirchhoff type:

$$\begin{cases} u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \displaystyle\int_0^t g_1(t-s)\Delta u(s)\mathrm{d}s - \Delta u_t = f_1(u,v), & (x,t) \in \Omega \times (0,\infty), \\ v_{tt} - M(\|\nabla v\|_2^2)\Delta v + \displaystyle\int_0^t g_2(t-s)\Delta v(s)\mathrm{d}s - \Delta v_t = f_2(u,v), & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, \quad v(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$, M is a positive C^1 function and $g_i(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, $f_i(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}^2 (i = 1, 2)$ are given functions to be specified later.

The motivation of our work is due to some results regarding viscoelastic wave equations of Kirchhoff type. The single wave equation of the form

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + h(u_t) = f(u), \quad (x,t) \in \Omega \times (0,\infty), \tag{1.2}$$

is a model to describe the motion of deformable solids as hereditary effect is incorporated. Equation (1.2) was first studied by Torrejón and Young [24] who proved the existence of weakly asymptotic stable solution for large analytical datum. Later, Rivera [12] showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions. Then, Wu and Tsai [26] treated equation (1.2) for $h(u_t) = -\Delta u_t$, they established the global existence as well as energy decay under assumption on the nonnegative kernel $g'(t) \leq -rg(t), \forall t \geq 0$ for some $r > 0$. This energy decay result was recently improved by Wu

in [29] under a weaker condition on g (i.e., $g'(t) \leq 0$ for $t \geq 0$). For a single wave equation of Kirchhoff type that without the viscoelastic term, we refer the reader to Refs. [15, 16, 17, 18, 19].

The system of wave equations that without viscoelastic terms ($i.e., g_i = 0, i = 1, 2$) has also been extensively studied and many results concerning local, global existence, decay and blow-up have been established. For example, Park and Bae [20] considered the system of wave equations with nonlinear dampings for $f_1(u, v) = \mu |u|^{q-1} u$ and $f_2(u, v) = \mu |v|^{q-1} v$, $q \geq 1$, and showed the global existence and asymptotic behavior of solutions under some restriction on the initial energy. Later, Benaissa and Messaoudi [2] discussed blow-up properties for negative initial energy. Recently, Wu and Tsai [27] studied the system (1.1) for $g_i = 0$ $(i = 1, 2)$. Under some suitable assumptions on f_i $(i = 1, 2)$, they proved the local existence of solutions by Banach fixed point theorem and blow-up of solutions by using the method of Li and Tsai in [5], where three different cases on the sign of the initial energy $E(0)$ are considered.

For the case of $M \equiv 1$ and in the presence of the viscoelastic terms ($i.e., g_i \neq 0, i = 1, 2$), problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g_1(t-\tau)\Delta u(\tau) \mathrm{d}\tau - \Delta u_t = f_1(u, v), & (x, t) \in \Omega \times (0, T), \\ v_{tt} - \Delta v + \displaystyle\int_0^t g_2(t-\tau)\Delta v(\tau) \mathrm{d}\tau - \Delta v_t = f_2(u, v), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, \quad v(x, t) = 0, & (x, t) \in \partial\Omega \times [0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.3}$$

was recently studied by Liang and Gao in [11]. Under suitable assumptions on the functions g_i, f_i $(i = 1, 2)$ and certain initial data in the stable set, they proved that the decay rate of the solution energy is exponential. Conversely, for certain initial data in the unstable set, there are solutions with positive initial energy that blow up in finite time.

It is also worth mentioning the work [4] in which the authors considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g_1(t-\tau)\Delta u(\tau) \mathrm{d}\tau + |u_t|^{m-1} u_t = f_1(u, v), & (x, t) \in \Omega \times (0, T), \\ v_{tt} - \Delta v + \displaystyle\int_0^t g_2(t-\tau)\Delta v(\tau) \mathrm{d}\tau + |v_t|^{\gamma-1} v_t = f_2(u, v), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, \quad v(x, t) = 0, & (x, t) \in \partial\Omega \times [0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n, n = 1, 2, 3$. Under suitable assumptions on the functions g_i, f_i $(i = 1, 2)$, the initial data and the parameters in the above problem, they established local existence, global existence and blow-up property (the initial energy $E(0) < 0$). This latter blow-up result has been improved by Messaoudi and Said-Houari [14], into

certain solutions with positive initial energy. For other papers related to existence, uniform decay and blow-up of solutions of nonlinear wave equations, see [1, 3, 6, 7, 8, 9, 13, 21, 22, 23, 25, 28, 29, 30] and references therein.

Motivated by the above mentioned research, we consider in the present work the coupled system (1.1) with nonzero g_i $(i = 1, 2)$ and nonconstant $M(s)$. We show that, under suitable assumptions on the functions g_i, f_i $(i = 1, 2)$ and certain initial conditions, the solutions are global in time and the energy decays exponentially.

This paper is organized as follows. In section 2, we first give some assumptions, notations and lemmas which will be used later, and then state the local existence. In section 3, we devote to state and prove our main result.

2 Preliminaries and main result

In this section we first present some assumptions, notations and lemmas needed for our work, and then state the local existence theorem. First, we make the following assumptions:

(A1) $M(s)$ is a positive C^1 function for $s \geq 0$ satisfying

$$M(s) = m_0 + m_1 s^\gamma, \quad m_0 > 0, \quad m_1 \geq 0 \quad \text{and} \quad \gamma \geq 1.$$

(A2) $g_i(t) : \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1, 2)$ belong to $C^1(\mathbb{R}^+)$ and satisfy

$$g_i(t) \geq 0, \quad g_i'(t) \leq 0, \quad \text{for} \quad t \geq 0.$$

Next, we introduce some notations:

$$k_i = m_0 - \int_0^\infty g_i(s)\mathrm{d}s > 0, \quad k = \min\{k_1, k_2\},$$

$$(g_i \circ \nabla w)(t) = \int_0^t g_i(t-s)\|\nabla w(t) - \nabla w(s)\|_2^2 \,\mathrm{d}s,$$

$$g_0 = \max\{g_1, g_2\}, \quad \|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}, \quad 1 \leq q \leq \infty,$$

the Hilbert space $L^2(\Omega)$ endowed with the inner product

$$(u, v) = \int_\Omega u(x)v(x)\mathrm{d}x,$$

and the functions $f_1(u, v)$ and $f_2(u, v)$ (see also [14])

$$f_1(u, v) = \left[a|u + v|^{2(p+1)}(u + v) + b|u|^p u|v|^{p+2}\right],$$

$$f_2(u, v) = \left[a|u + v|^{2(p+1)}(u + v) + b|u|^{p+2}|v|^p v\right],$$

where $a, b > 0$ are constants and p satisfies

$$\begin{cases} -1 < p, & \text{if} \quad n = 1, 2, \\ -1 < p \leq (3-n)/(n-2), & \text{if} \quad n \geq 3. \end{cases} \tag{2.1}$$

One can easily verify that

$$uf_1(u,v) + vf_2(u,v) = 2(p+2)F(u,v), \quad \forall\ (u,v) \in \mathbb{R}^2,$$

where

$$F(u,v) = \frac{1}{2(p+2)}\left[a|u+v|^{2(p+2)} + 2b|uv|^{p+2}\right].$$

The energy functional $E(t)$ and auxiliary functional $I(t)$, $J(t)$ of the solutions $(u(t), v(t))$ of problem (1.1) are defined as follows

$$\begin{aligned} I(t) :=& I(u(t),v(t)) = \left(m_0 - \int_0^t g_1(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + \left(m_0 - \int_0^t g_2(s)\mathrm{d}s\right)\|\nabla v(t)\|_2^2 \\ &+ (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) - 2(p+2)\int_\Omega F(u,v)\mathrm{d}x, \end{aligned} \tag{2.2}$$

$$\begin{aligned} J(t) :=& J(u(t),v(t)) = \frac{1}{2}\left[\left(m_0 - \int_0^t g_1(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^t g_2(s)\,\mathrm{d}s\right)\|\nabla v\|_2^2\right] - \int_\Omega F(u,v)\mathrm{d}x \\ &+ \frac{1}{2}\left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) + \frac{m_1}{\gamma+1}\left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}\right)\right], \end{aligned} \tag{2.3}$$

$$E(t) := E(u(t),v(t)) = \frac{1}{2}(\|u_t\|_2^2 + \|v_t\|_2^2) + J(u(t),v(t)). \tag{2.4}$$

As in [29], we can get

$$\begin{aligned} E'(t) =& -(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) - \frac{1}{2}[g_1(t)\|\nabla u\|_2^2 + g_2(t)\|\nabla v\|_2^2] + \frac{1}{2}[(g_1' \circ \nabla u)(t) + (g_2' \circ \nabla v)(t)] \\ &\leq 0, \quad \forall\ t \geq 0. \end{aligned} \tag{2.5}$$

Then we have

$$E(t) + \int_s^t (\|\nabla u_t(\tau)\|_2^2 + \|\nabla v_t(\tau)\|_2^2)\mathrm{d}\tau \leq E(s), \tag{2.6}$$

for $0 \leq s \leq t \leq T$.

Then, we give two lemmas which will be used throughout this work.

Lemma 2.1 *(Sobolev-Poincaré inequality [15]) If $2 \leq p \leq \frac{2N}{N-2}$, then*

$$\|u\|_p \leq C_p\|\nabla u\|_{2},$$

for $u \in H_0^1(\Omega)$ holds with some constants C_p.

Lemma 2.2 *(See [10]) Let $h : [0, \infty) \to [0, \infty)$ be a non-increasing function and assume that there exists a constant $r > 0$ such that*

$$\int_t^\infty h(s)\mathrm{d}s \le rh(t), \quad \forall\, t \in [0, \infty).$$

Then we have

$$h(t) \le h(0)e^{1-\frac{t}{r}}, \quad \forall\, t \ge r.$$

We now state a local existence theorem for system (1.1), whose proof follows the arguments in [26, 27]:

Theorem 2.3 *Suppose that (2.1), (A1) and (A2) hold, and that $u_0, v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1, v_1 \in L^2(\Omega)$. Then problem (1.1) has a unique local solution*

$$u, v \in C([0, T]; H_0^1(\Omega) \cap H^2(\Omega)), \quad u_t, v_t \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega)),$$

for some $T > 0$. Moreover, at least one of the following statements is valid:

(1) $T = \infty$,

(2) $\lim\limits_{t\to T^-} (\|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta v\|_2^2) = \infty.$

3 Global existence and energy decay

In this section, we consider the global existence and energy decay of solutions for problem (1.1). We first introduce two lemmas which are essential in our proof.

Lemma 3.1 ([14, Lemma 3.2]) *Assume that (2.1) holds. Then there exists $\eta > 0$ such that for any $(u, v) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega))$, we have*

$$\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \le \eta(k_1\|\nabla u\|_2^2 + k_2\|\nabla v\|_2^2)^{p+2}. \tag{3.1}$$

In order to prove our result and for the sake of simplicity, we take $a = b = 1$ and introduce

$$B = \eta^{\frac{1}{2(p+2)}}, \quad \alpha_* = B^{-\frac{p+2}{p+1}}, \quad E_1 = \left(\frac{1}{2} - \frac{1}{2(p+2)}\right)\alpha_*^2, \tag{3.2}$$

where η is the optimal constant in (3.1). Next, we will state and prove a lemma which similar to the one introduced firstly by Vitillaro in [25] to study a class of a single wave equation.

Lemma 3.2 *Suppose that (2.1), (A1) and (A2) hold. Let (u, v) be the solution of system (1.1). Assume further that $E(0) < E_1$ and*

$$(k_1\|\nabla u_0\|_2^2 + k_2\|\nabla v_0\|_2^2)^{1/2} < \alpha_*. \tag{3.3}$$

Then

$$\left(k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right)^{1/2} < \alpha_*, \tag{3.4}$$

for all $t \in [0, T)$.

Proof. We first note that, by (2.4), (3.1) and the definition of B, we have

$$\begin{aligned}
E(t) \geq & \frac{1}{2}\left[k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] \\
& - \frac{1}{2(p+2)}\left(\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right) \\
\geq & \frac{1}{2}\left[k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] \\
& - \frac{B^{2(p+2)}}{2(p+2)}\left(k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2\right)^{p+2} \\
\geq & \frac{1}{2}\alpha^2 - \frac{B^{2(p+2)}}{2(p+2)}\alpha^{2(p+2)} \triangleq G(\alpha),
\end{aligned} \tag{3.5}$$

where

$$\alpha = \left(k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right)^{1/2}.$$

It is easy to verify that $G(\alpha)$ is increasing in $(0, \alpha_*)$, decreasing in (α_*, ∞), and that $G(\alpha) \to -\infty$, as $\alpha \to \infty$, and

$$G(\alpha)_{\max} = G(\alpha_*) = \frac{1}{2}\alpha_*^2 - \frac{B^{2(p+2)}}{2(p+2)}\alpha_*^{2(p+2)} = E_1,$$

where α_* is given in (3.2).

Now we establish (3.4) by contradiction. First we assume that (3.4) is not true over $[0, T)$, then it follows from the continuity of $(u(t), v(t))$ that there exists $t_0 \in (0, T)$ such that

$$\left(k_1\|\nabla u(t_0)\|_2^2 + k_2\|\nabla v(t_0)\|_2^2 + (g_1 \circ \nabla u)(t_0) + (g_2 \circ \nabla v)(t_0)\right)^{1/2} = \alpha_*.$$

By (3.5), we see that

$$\begin{aligned}
E(t_0) &\geq G\left[\left(k_1\|\nabla u(t_0)\|_2^2 + k_2\|\nabla v(t_0)\|_2^2 + (g_1 \circ \nabla u)(t_0) + (g_2 \circ \nabla v)(t_0)\right)^{1/2}\right] \\
&= G(\alpha_*) = E_1.
\end{aligned}$$

This is impossible since $E(t) \leq E(0) < E_1$, $t \geq 0$. Hence (3.4) is established. $\Box$

Now we are in a position to state and prove our main result.

Theorem 3.3 (Global existence and energy decay) *Assume that* (2.1), (*A1*) *and* (*A2*) *hold. If the initial data* $(u_0, u_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(\Omega)$, $(v_0, v_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(\Omega)$, *satisfy* $E(0) < E_1$ *and*

$$(k_1\|\nabla u_0\|_2^2 + k_2\|\nabla v_0\|_2^2)^{1/2} < \alpha_*, \tag{3.6}$$

where the constants α_*, E_1 *are defined in* (3.2), *then the corresponding solution to system* (1.1) *globally exists, i.e.,* $T = \infty$.

Moreover, if $m_0 - k > 0$ *is sufficiently small such that*

$$1 - \eta\left(\frac{2(p+2)}{p+1}E(0)\right)^{p+1} - \frac{5(m_0-k)(p+2)}{2k(p+1)} > 0, \tag{3.7}$$

then we have the following decay estimates

$$E(t) \leq E(0)e^{1-\lambda C_0^{-1}t}$$

for every $t \geq \lambda C_0^{-1}$, *where* C_0 *is some positive constant.*

Proof. First, we prove that $T = \infty$. Since $E(0) < E_1$ and

$$(k_1\|\nabla u_0\|_2^2 + k_2\|\nabla v_0\|_2^2)^{1/2} < \alpha_*,$$

it follows from Lemma 3.2 that

$$\begin{aligned}
&k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 \\
\leq &k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) \\
< &\alpha_*^2 = \eta^{-\frac{1}{p+1}},
\end{aligned}$$

which implies that

$$\begin{aligned}
I(t) \geq &\left(m_0 - \int_0^\infty g_1(s)\mathrm{d}s\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^\infty g_2(s)\mathrm{d}s\right)\|\nabla v\|_2^2 \\
&+ (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) - 2(p+2)\int_\Omega F(u,v)\mathrm{d}x \\
\geq &k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 - 2(p+2)\int_\Omega F(u,v)\mathrm{d}x \\
= &k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 - \left(\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right) \\
\geq &k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 - \eta\left(k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2\right)^{p+2} \\
= &(k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2)\left[1 - \eta\left(k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2\right)^{p+1}\right] \geq 0,
\end{aligned}$$

for $t \in [0,T)$, where we have used (3.1). Further, by (2.2) and (2.3), we have

$$\begin{aligned}
J(t) \geq &\frac{1}{2}\left[\left(m_0 - \int_0^t g_1(s)\mathrm{d}s\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^t g_2(s)\mathrm{d}s\right)\|\nabla v\|_2^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] \\
&- \int_\Omega F(u,v)\mathrm{d}x - \frac{1}{2(p+2)}I(t) + \frac{1}{2(p+2)}I(t) \\
\geq &\left(\frac{1}{2} - \frac{1}{2(p+2)}\right)\left[\left(m_0 - \int_0^t g_1(s)\mathrm{d}s\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^t g_2(s)\mathrm{d}s\right)\|\nabla v\|_2^2\right] \\
&+ \left(\frac{1}{2} - \frac{1}{2(p+2)}\right)[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] + \frac{1}{2(p+2)}I(t) \\
\geq &\frac{p+1}{2(p+2)}\left[k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] + \frac{1}{2(p+2)}I(t) \\
\geq &0,
\end{aligned} \tag{3.8}$$

from (3.8) and (2.4), we deduce that

$$k\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) \leq k_1\|\nabla u(t)\|_2^2 + k_2\|\nabla v(t)\|_2^2 \leq \frac{2(p+2)}{p+1}J(t) \leq \frac{2(p+2)}{p+1}E(t). \tag{3.9}$$

Multiplying the first equation of system (1.1) by $-2\Delta u$, and integrating it over Ω, we get

$$\begin{aligned}&\frac{d}{dt}\left\{\|\Delta u\|_2^2-2\int_\Omega u_t\Delta u\mathrm{d}x\right\}+2M(\|\nabla u\|_2^2)\|\Delta u\|_2^2\\ \le&2\|\nabla u_t\|_2^2-2\int_\Omega f_1(u,v)\Delta u\mathrm{d}x+2\int_0^t g_1(t-s)\int_\Omega \Delta u(s)\Delta u(t)\mathrm{d}x\mathrm{d}s\end{aligned}$$

By Young's inequality, we have

$$2\int_0^t g_1(t-s)\int_\Omega \Delta u(s)\Delta u(t)\mathrm{d}x\mathrm{d}s\le 2\theta\|\Delta u(t)\|_2^2+\frac{\|g_1\|_{L^1}}{2\theta}\int_0^t g_1(t-s)\|\Delta u(s)\|_2^2\mathrm{d}s.$$

Thus, we obtain

$$\begin{aligned}&\frac{d}{dt}\left\{\|\Delta u\|_2^2-2\int_\Omega u_t\Delta u\mathrm{d}x\right\}+\left[2M(\|\nabla u\|_2^2)-2\theta\right]\|\Delta u\|_2^2\\ \le&2\|\nabla u_t\|_2^2+\frac{\|g_1\|_{L^1}}{2\theta}\int_0^t g_1(t-s)\|\Delta u(s)\|_2^2\mathrm{d}s-2\int_\Omega f_1(u,v)\Delta u\mathrm{d}x,\end{aligned}\tag{3.10}$$

where $0<\theta\le\frac{\|g_0\|_{L^1}}{2}$. Similarly, we also have

$$\begin{aligned}&\frac{d}{dt}\left\{\|\Delta v\|_2^2-2\int_\Omega v_t\Delta v\mathrm{d}x\right\}+\left[2M(\|\nabla v\|_2^2)-2\theta\right]\|\Delta v\|_2^2\\ \le&2\|\nabla v_t\|_2^2+\frac{\|g_2\|_{L^1}}{2\theta}\int_0^t g_2(t-s)\|\Delta v(s)\|_2^2\mathrm{d}s-2\int_\Omega f_2(u,v)\Delta v\mathrm{d}x.\end{aligned}\tag{3.11}$$

Now, combining (3.10) with (3.11), we get

$$\begin{aligned}&\frac{d}{dt}\left\{\|\Delta u\|_2^2-2\int_\Omega u_t\Delta u\mathrm{d}x+\|\Delta v\|_2^2-2\int_\Omega v_t\Delta v\mathrm{d}x\right\}\\ &+2\left[M(\|\nabla u\|_2^2)-\theta\right]\|\Delta u\|_2^2+2\left[M(\|\nabla v\|_2^2)-\theta\right]\|\Delta v\|_2^2\\ \le&2(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2)+\frac{\|g_1\|_{L^1}}{2\theta}\int_0^t g_1(t-s)\|\Delta u(s)\|_2^2\mathrm{d}s\\ &+\frac{\|g_2\|_{L^1}}{2\theta}\int_0^t g_2(t-s)\|\Delta v(s)\|_2^2\mathrm{d}s-2\int_\Omega (f_1(u,v)\Delta u+f_2(u,v)\Delta v)\mathrm{d}x.\end{aligned}\tag{3.12}$$

Multiplying (3.12) by ε, $0<\varepsilon\le1$, and multiplying (2.5) by δ, δ is some positive constant, and then summing up, we deduce

$$\begin{aligned}&\frac{d}{dt}E^*(t)+(\delta-2\varepsilon)(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2)+2\varepsilon\left[M(\|\nabla u\|_2^2)-\theta\right]\|\Delta u\|_2^2+2\varepsilon\left[M(\|\nabla v\|_2^2)-\theta\right]\|\Delta v\|_2^2\\ \le&\varepsilon\frac{\|g_1\|_{L^1}}{2\theta}\int_0^t g_1(t-s)\|\Delta u(s)\|_2^2\mathrm{d}s+\varepsilon\frac{\|g_2\|_{L^1}}{2\theta}\int_0^t g_2(t-s)\|\Delta v(s)\|_2^2\mathrm{d}s\\ &-2\varepsilon\int_\Omega (f_1(u,v)\Delta u+f_2(u,v)\Delta v)\mathrm{d}x,\end{aligned}\tag{3.13}$$

where

$$E^*(t)=\delta E(t)+\varepsilon(\|\Delta u\|_2^2+\|\Delta v\|_2^2)-2\varepsilon\left(\int_\Omega u_t\Delta u\mathrm{d}x+\int_\Omega v_t\Delta v\mathrm{d}x\right).$$

By Young's inequality, we get

$$\left|2\varepsilon\int_\Omega u_t\Delta u\mathrm{d}x\right|\le 2\varepsilon\|u_t\|_2^2+\frac{\varepsilon}{2}\|\Delta u\|_2^2$$

and

$$\left|2\varepsilon \int_\Omega v_t \Delta v \mathrm{d}x\right| \le 2\varepsilon \|v_t\|_2^2 + \frac{\varepsilon}{2}\|\Delta v\|_2^2.$$

Noting that $J(t) \ge 0$ by (3.8), then, by (2.4), we have

$$\|u_t\|_2^2 + \|v_t\|_2^2 \le 2E(t). \tag{3.14}$$

Therefore, choosing $\delta = 5\varepsilon$, we observe that

$$E^*(t) \ge \frac{\varepsilon}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta v\|_2^2\right). \tag{3.15}$$

Moreover, by Hölder's inequality, Lemma 2.1 and (3.9), we see that

$$\begin{aligned}
&\left|\int_\Omega f_1(u,v)\Delta u \mathrm{d}x\right| \\
\le& \int_\Omega \left(|u+v|^{2p+3} + |u|^{p+1}|v|^{p+2}\right)\Delta u \mathrm{d}x \\
\le& C\int_\Omega \left(|u|^{2p+3} + |v|^{2p+3} + |u|^{p+1}|v|^{p+2}\right)\Delta u \mathrm{d}x \\
\le& C\left[\int_\Omega \left(|u|^{2p+3} + |v|^{2p+3} + |u|^{p+1}|v|^{p+2}\right)^2 \mathrm{d}x\right]^{\frac{1}{2}} \|\Delta u\|_2 \\
\le& C\left[\int_\Omega \left(|u|^{2(2p+3)} + |v|^{2(2p+3)} + |u|^{2(p+1)}|v|^{2(p+2)}\right) \mathrm{d}x\right]^{\frac{1}{2}} \|\Delta u\|_2 \\
\le& C\left[\|u\|_{2(2p+3)}^{2p+3} + \|v\|_{2(2p+3)}^{2p+3} + \|u\|_{2(2p+3)}^{p+1}\|v\|_{2(2p+3)}^{p+2}\right]\|\Delta u\|_2 \\
\le& C\left[\|\nabla u\|_2^{2p+3} + \|\nabla v\|_2^{2p+3} + \|\nabla u\|_2^{p+1}\|\nabla v\|_2^{p+2}\right]\|\Delta u\|_2 \\
\le& C_*\|\Delta u\|_2,
\end{aligned} \tag{3.16}$$

here we denote by $C > 0$ a generic constant that may vary even from line to line within the same formula and $C_* = 3C\left(\frac{2(p+2)}{k(p+1)}E(0)\right)^{(2p+3)/2}$. Similarly, we have

$$\left|\int_\Omega f_2(u,v)\Delta v \mathrm{d}x\right| \le C_*\|\Delta v\|_2. \tag{3.17}$$

Combining (3.15)–(3.17), we deduce

$$2\varepsilon\left|\int_\Omega f_1(u,v)\Delta u \mathrm{d}x + \int_\Omega f_2(u,v)\Delta v \mathrm{d}x\right| \le 2\varepsilon C_*(\|\Delta u\|_2 + \|\Delta v\|_2) \le 4\sqrt{2\varepsilon}C_* E^*(t)^{\frac{1}{2}}. \tag{3.18}$$

Substituting (3.18) into (3.13), and then integrating it over $(0,t)$, we obtain

$$\begin{aligned}
&E^*(t) + 2\varepsilon\left(m_0 - \theta - \frac{\|g_0\|_{L^1}^2}{4\theta}\right)\int_0^t (\|\Delta u(s)\|_2^2 + \|\Delta v(s)\|_2^2)\mathrm{d}s \\
\le& E^*(0) + 4\sqrt{2\varepsilon}C_*\int_0^t E^*(s)^{\frac{1}{2}}\mathrm{d}s.
\end{aligned} \tag{3.19}$$

Taking $\theta = \frac{\|g_0\|_{L^1}}{2}$ in (3.19), and then by Gronwall's Lemma, we deduce

$$E^*(t) \le \left(2\sqrt{2\varepsilon}C_* t + E^*(0)^{\frac{1}{2}}\right)^2,$$

for any $t \geq 0$. Therefore, by (3.15) and Theorem 2.3, we have $T = \infty$ as long as ε is fixed.

Next, we want to derive the decay rate of energy function for system (1.1). Multiplying the first equation of system (1.1) by u and the second equation of system (1.1) by v, integrating them over $\Omega \times [t_1, t_2](0 \leq t_1 \leq t_2)$, using integration by parts and summing up, we have

$$\begin{aligned}
&\int_\Omega u_t u \mathrm{d}x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \|u_t\|_2^2 \mathrm{d}t + \int_\Omega v_t v \mathrm{d}x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \|v_t\|_2^2 \mathrm{d}t + \int_{t_1}^{t_2} M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 \mathrm{d}t \\
&+ \int_{t_1}^{t_2} M(\|\nabla v\|_2^2)\|\nabla v\|_2^2 \mathrm{d}t + \int_{t_1}^{t_2} \int_\Omega \int_0^t g_1(t-s)\Delta u(s) u(t) \mathrm{d}s \mathrm{d}x \mathrm{d}t \\
&+ \int_{t_1}^{t_2} \int_\Omega \int_0^t g_2(t-s)\Delta v(s) v(t) \mathrm{d}s \mathrm{d}x \mathrm{d}t + \int_{t_1}^{t_2} \int_\Omega \nabla u_t \nabla u \mathrm{d}x \mathrm{d}t + \int_{t_1}^{t_2} \int_\Omega \nabla v_t \nabla v \mathrm{d}x \mathrm{d}t \\
=& \int_{t_1}^{t_2} \int_\Omega [f_1(u,v)u + f_2(u,v)v] \mathrm{d}x \mathrm{d}t.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{t_1}^{t_2} M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 \mathrm{d}t + \int_{t_1}^{t_2} M(\|\nabla v\|_2^2)\|\nabla v\|_2^2 \mathrm{d}t - 2(p+2)\int_{t_1}^{t_2} \int_\Omega F(u,v) \mathrm{d}x \mathrm{d}t \\
=& - \int_\Omega u_t u \mathrm{d}x\Big|_{t_1}^{t_2} - \int_\Omega v_t v \mathrm{d}x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_\Omega \nabla u_t \nabla u \mathrm{d}x \mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega \nabla v_t \nabla v \mathrm{d}x \mathrm{d}t + \int_{t_1}^{t_2} \|u_t\|_2^2 \mathrm{d}t \\
&+ \int_{t_1}^{t_2} \|v_t\|_2^2 \mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega \int_0^t g_1(t-s)\Delta u(s) u(t) \mathrm{d}s \mathrm{d}x \mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega \int_0^t g_2(t-s)\Delta v(s) v(t) \mathrm{d}s \mathrm{d}x \mathrm{d}t.
\end{aligned}$$

It follows from (2.4) that

$$\begin{aligned}
&2\int_{t_1}^{t_2} E(t) \mathrm{d}t + \int_{t_1}^{t_2} \left[M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 + M(\|\nabla v\|_2^2)\|\nabla v\|_2^2\right] \mathrm{d}t - 2(p+2)\int_{t_1}^{t_2} \int_\Omega F(u,v) \mathrm{d}x \mathrm{d}t \\
=& 2\int_{t_1}^{t_2} (\|u_t\|_2^2 + \|v_t\|_2^2) \mathrm{d}t + \int_{t_1}^{t_2} \left[\left(m_0 - \int_0^t g_1(s) \mathrm{d}s\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^t g_2(s) \mathrm{d}s\right)\|\nabla v\|_2^2\right] \mathrm{d}t \\
&- 2\int_{t_1}^{t_2} \int_\Omega F(u,v) \mathrm{d}x \mathrm{d}t - \int_\Omega u_t u \mathrm{d}x\Big|_{t_1}^{t_2} - \int_\Omega v_t v \mathrm{d}x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_\Omega (\nabla u_t \nabla u + \nabla v_t \nabla v) \mathrm{d}x \mathrm{d}t \\
&- \int_{t_1}^{t_2} \int_\Omega \int_0^t g_1(t-s)\Delta u(s) u(t) \mathrm{d}s \mathrm{d}x \mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega \int_0^t g_2(t-s)\Delta v(s) v(t) \mathrm{d}s \mathrm{d}x \mathrm{d}t \\
&+ \int_{t_1}^{t_2} \left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) + \frac{m_1}{\gamma+1}\left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}\right)\right] \mathrm{d}t,
\end{aligned}$$

and then we arrive at

$$\begin{aligned}
&2\int_{t_1}^{t_2} E(t) \mathrm{d}t - 2(p+1)\int_{t_1}^{t_2} \int_\Omega F(u,v) \mathrm{d}x \mathrm{d}t \\
\leq& - \int_{t_1}^{t_2} \left[\int_0^t g_1(s) \mathrm{d}s \|\nabla u\|_2^2 + \int_0^t g_2(s) \mathrm{d}s \|\nabla v\|_2^2\right] \mathrm{d}t + \int_{t_1}^{t_2} \left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] \mathrm{d}t \\
&- \int_\Omega (u_t u + v_t v) \mathrm{d}x\Big|_{t_1}^{t_2} + 2\int_{t_1}^{t_2} (\|u_t\|_2^2 + \|v_t\|_2^2) \mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega (\nabla u_t \nabla u + \nabla v_t \nabla v) \mathrm{d}x \mathrm{d}t \\
&- \int_{t_1}^{t_2} \int_\Omega \int_0^t g_1(t-s)\Delta u(s) u(t) \mathrm{d}s \mathrm{d}x \mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega \int_0^t g_2(t-s)\Delta v(s) v(t) \mathrm{d}s \mathrm{d}x \mathrm{d}t. \qquad (3.20)
\end{aligned}$$

For the sixth term on the right-hand side of (3.20), we have

$$
\begin{aligned}
&-\int_\Omega\int_0^t g_1(t-s)\Delta u(s)u(t)\mathrm{d}s\mathrm{d}x\mathrm{d}t=\int_\Omega\int_0^t g_1(t-s)\nabla u(s)\nabla u(t)\mathrm{d}s\mathrm{d}x\\
=&\frac{1}{2}\left[\int_0^t g_1(t-s)\|\nabla u(t)\|_2^2\mathrm{d}s+\int_0^t g_1(t-s)\|\nabla u(s)\|_2^2\mathrm{d}s-(g_1\circ\nabla u)(t)\right].
\end{aligned}\tag{3.21}
$$

Similarly,

$$
\begin{aligned}
&-\int_\Omega\int_0^t g_2(t-s)\Delta v(s)v(t)\mathrm{d}s\mathrm{d}x\mathrm{d}t\\
=&\frac{1}{2}\left[\int_0^t g_2(t-s)\|\nabla v(t)\|_2^2\mathrm{d}s+\int_0^t g_2(t-s)\|\nabla v(s)\|_2^2\mathrm{d}s-(g_2\circ\nabla v)(t)\right].
\end{aligned}\tag{3.22}
$$

By (3.20), (3.21) and (3.22), we get

$$
\begin{aligned}
&2\int_{t_1}^{t_2}E(t)\mathrm{d}t-2(p+1)\int_{t_1}^{t_2}\int_\Omega F(u,v)\mathrm{d}x\mathrm{d}t\\
\le&-\frac{1}{2}\int_{t_1}^{t_2}\left[\int_0^t g_1(s)\mathrm{d}s\|\nabla u(t)\|_2^2+\int_0^t g_2(s)\mathrm{d}s\|\nabla v(t)\|_2^2\right]\mathrm{d}t+\frac{1}{2}\int_{t_1}^{t_2}\left[(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right]\mathrm{d}t\\
&-\int_\Omega(u_tu+v_tv)\mathrm{d}x\Big|_{t_1}^{t_2}+2\int_{t_1}^{t_2}(\|u_t\|_2^2+\|v_t\|_2^2)\mathrm{d}t-\int_{t_1}^{t_2}\int_\Omega(\nabla u_t\nabla u+\nabla v_t\nabla v)\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\int_{t_1}^{t_2}\int_0^t g_1(t-s)\|\nabla u(s)\|_2^2\mathrm{d}s\mathrm{d}t+\frac{1}{2}\int_{t_1}^{t_2}\int_0^t g_2(t-s)\|\nabla v(s)\|_2^2\mathrm{d}s\mathrm{d}t\\
\le&-\int_\Omega(u_tu+v_tv)\mathrm{d}x\Big|_{t_1}^{t_2}+2\int_{t_1}^{t_2}(\|u_t\|_2^2+\|v_t\|_2^2)\mathrm{d}t-\int_{t_1}^{t_2}\int_\Omega(\nabla u_t\nabla u+\nabla v_t\nabla v)\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\int_{t_1}^{t_2}\left[\int_0^t g_1(t-s)\|\nabla u(s)\|_2^2\mathrm{d}s+\int_0^t g_2(t-s)\|\nabla v(s)\|_2^2\mathrm{d}s\right]\mathrm{d}t\\
&+\frac{1}{2}\int_{t_1}^{t_2}\left[(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right]\mathrm{d}t\\
=&I_1+I_2-I_3+I_4+I_5.
\end{aligned}\tag{3.23}
$$

In what follows we will estimate $I_1,\dots,I_5$ in (3.23). Firstly, by Hölder's inequality, Young's inequality and Lemma 2.1, we have

$$
\begin{aligned}
&\int_\Omega|u(t)u_t(t)|\mathrm{d}x+\int_\Omega|v(t)v_t(t)|\mathrm{d}x\\
\le&\frac{1}{2}\|u(t)\|_2^2+\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|v(t)\|_2^2+\frac{1}{2}\|v_t(t)\|_2^2\\
\le&\frac{C_p^2}{2}\|\nabla u(t)\|_2^2+\frac{C_p^2}{2}\|\nabla v(t)\|_2^2+\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|v_t(t)\|_2^2.
\end{aligned}
$$

It also follows from (3.9), (2.4) and (2.5) that $k(\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2)\le\frac{2(p+2)}{p+1}E(t)$, $\|u_t(t)\|_2^2+\|v_t(t)\|_2^2\le 2E(t)$ and $E(t)$ is a non-increasing function. Therefore, we have

$$
I_1\le\int_\Omega|u(t)u_t(t)|\mathrm{d}x\Big|_{t_1}^{t_2}+\int_\Omega|v(t)v_t(t)|\mathrm{d}x\Big|_{t_1}^{t_2}\le 2C_1E(t_1),\tag{3.24}
$$

where $C_1 = \frac{p+2}{k(p+1)}C_p^2 + 1$.

For I_2 in (3.23), applying $\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \le -E'(t)$ from (2.5), we have

$$I_2 \le 2C_p^2 \int_{t_1}^{t_2} (\|\nabla u_t(t)\|_2^2 + \|\nabla v_t(t)\|_2^2)\mathrm{d}t \le 2C_p^2 E(t_1), \tag{3.25}$$

We also have the following estimate

$$\begin{aligned}
I_3 =& \int_{t_1}^{t_2} \int_\Omega \nabla u(t) \nabla u_t(t) \mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} \int_\Omega \nabla v(t) \nabla v_t(t) \mathrm{d}x\mathrm{d}t \\
=& \frac{1}{2} \int_{t_1}^{t_2} \frac{d}{\mathrm{d}t} \|\nabla u(t)\|_2^2 \mathrm{d}t + \frac{1}{2} \int_{t_1}^{t_2} \frac{d}{\mathrm{d}t} \|\nabla v(t)\|_2^2 \mathrm{d}t \\
=& \frac{1}{2} \left(\|\nabla u(t_2)\|_2^2 - \|\nabla u(t_1)\|_2^2 \right) + \frac{1}{2} \left(\|\nabla v(t_2)\|_2^2 - \|\nabla v(t_1)\|_2^2 \right) \\
\le& \frac{2(p+2)}{k(p+1)} E(t_1) \triangleq C_3 E(t_1).
\end{aligned} \tag{3.26}$$

To estimate I_4, using Young's inequality for convolution $\|\phi * \psi\|_q \le \|\phi\|_r \|\psi\|_s$, with $\frac{1}{q} = \frac{1}{r} + \frac{1}{s} - 1$, $1 \le q, r, s \le \infty$, noting that if $q = 1$, then $r = 1$ and $s = 1$, we get

$$\int_{t_1}^{t_2} \int_0^t g_1(t-s) \|\nabla u(s)\|_2^2 \mathrm{d}s\mathrm{d}t \le \int_{t_1}^{t_2} g_1(t)\mathrm{d}t \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 \mathrm{d}t \le (m_0 - k_1) \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 \mathrm{d}t, \tag{3.27}$$

and

$$\int_{t_1}^{t_2} \int_0^t g_2(t-s) \|\nabla v(s)\|_2^2 \mathrm{d}s\mathrm{d}t \le \int_{t_1}^{t_2} g_2(t)\mathrm{d}t \int_{t_1}^{t_2} \|\nabla v(t)\|_2^2 \mathrm{d}t \le (m_0 - k_2) \int_{t_1}^{t_2} \|\nabla v(t)\|_2^2 \mathrm{d}t. \tag{3.28}$$

Hence, by (3.9), (3.27) and (3.28), we obtain

$$\begin{aligned}
2I_4 =& \int_{t_1}^{t_2} \int_0^t g_1(t-s) \|\nabla u(s)\|_2^2 \mathrm{d}s\mathrm{d}t + \int_{t_1}^{t_2} \int_0^t g_2(t-s) \|\nabla v(s)\|_2^2 \mathrm{d}s\mathrm{d}t \\
\le& (m_0 - k_1) \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 \mathrm{d}t + (m_0 - k_2) \int_{t_1}^{t_2} \|\nabla v(t)\|_2^2 \mathrm{d}t \\
\le& (m_0 - k) \int_{t_1}^{t_2} (\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) \mathrm{d}t \\
\le& \frac{2(m_0 - k)(p+2)}{k(p+1)} \int_{t_1}^{t_2} E(t)\mathrm{d}t.
\end{aligned} \tag{3.29}$$

By virtue of (3.9), we also have

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_0^t g_1(t-s) \|\nabla u(t)\|_2^2 \mathrm{d}s\mathrm{d}t + \int_{t_1}^{t_2} \int_0^t g_2(t-s) \|\nabla v(t)\|_2^2 \mathrm{d}s\mathrm{d}t \\
\le& (m_0 - k_1) \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 \mathrm{d}t + (m_0 - k_2) \int_{t_1}^{t_2} \|\nabla v(t)\|_2^2 \mathrm{d}t \\
\le& (m_0 - k) \int_{t_1}^{t_2} (\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) \mathrm{d}t \\
\le& \frac{2(m_0 - k)(p+2)}{k(p+1)} \int_{t_1}^{t_2} E(t)\mathrm{d}t.
\end{aligned} \tag{3.30}$$

From (3.29) and (3.30), we see that

$$
\begin{aligned}
I_5 =& \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g_1(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2\mathrm{d}s\mathrm{d}t + \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g_2(t-s)\|\nabla v(t)-\nabla v(s)\|_2^2\mathrm{d}s\mathrm{d}t \\
\leq& \frac{1+2\epsilon}{2}\int_{t_1}^{t_2}\int_0^t g_1(t-s)\|\nabla u(t)\|_2^2\mathrm{d}s\mathrm{d}t + \frac{1}{2}\left(1+\frac{1}{2\epsilon}\right)\int_{t_1}^{t_2}\int_0^t g_1(t-s)\|\nabla u(s)\|_2^2\mathrm{d}s\mathrm{d}t \\
&+ \frac{1+2\epsilon}{2}\int_{t_1}^{t_2}\int_0^t g_2(t-s)\|\nabla v(t)\|_2^2\mathrm{d}s\mathrm{d}t + \frac{1}{2}\left(1+\frac{1}{2\epsilon}\right)\int_{t_1}^{t_2}\int_0^t g_2(t-s)\|\nabla v(s)\|_2^2\mathrm{d}s\mathrm{d}t \\
\leq& \frac{1+2\epsilon}{2}\left[\int_{t_1}^{t_2}\int_0^t g_1(t-s)\|\nabla u(t)\|_2^2\mathrm{d}s\mathrm{d}t + \int_{t_1}^{t_2}\int_0^t g_2(t-s)\|\nabla v(t)\|_2^2\mathrm{d}s\mathrm{d}t\right] \\
&+ \frac{1}{2}\left(1+\frac{1}{2\epsilon}\right)\left[\int_{t_1}^{t_2}\int_0^t g_1(t-s)\|\nabla u(s)\|_2^2\mathrm{d}s\mathrm{d}t + \int_{t_1}^{t_2}\int_0^t g_2(t-s)\|\nabla v(s)\|_2^2\mathrm{d}s\mathrm{d}t\right] \\
\leq& (1+2\epsilon)\frac{(m_0-k)(p+2)}{k(p+1)}\int_{t_1}^{t_2}E(t)\mathrm{d}t + \left(1+\frac{1}{2\epsilon}\right)\frac{(m_0-k)(p+2)}{k(p+1)}\int_{t_1}^{t_2}E(t)\mathrm{d}t \\
=& \left(2+2\epsilon+\frac{1}{2\epsilon}\right)\frac{(m_0-k)(p+2)}{k(p+1)}\int_{t_1}^{t_2}E(t)\mathrm{d}t,
\end{aligned}
\tag{3.31}
$$

where we have used the following Young's inequalities

$$
\left|2\int_\Omega \nabla u(t)\nabla u(s)\mathrm{d}x\right| \leq 2\epsilon\|\nabla u(t)\|_2^2 + \frac{1}{2\epsilon}\|\nabla u(s)\|_2^2, \quad \forall\ \epsilon>0,
$$

and

$$
\left|2\int_\Omega \nabla v(t)\nabla v(s)\mathrm{d}x\right| \leq 2\epsilon\|\nabla v(t)\|_2^2 + \frac{1}{2\epsilon}\|\nabla v(s)\|_2^2, \quad \forall\ \epsilon>0.
$$

Combining (3.23)–(3.31), we obtain

$$
\begin{aligned}
& 2\int_{t_1}^{t_2}E(t)\mathrm{d}t - 2(p+1)\int_{t_1}^{t_2}\int_\Omega F(u,v)\mathrm{d}x\mathrm{d}t \\
\leq& 2C_1E(t_1) + 2C_p^2E(t_1) + \left(2+2\epsilon+\frac{1}{2\epsilon}\right)\frac{(m_0-k)(p+2)}{k(p+1)}\int_{t_1}^{t_2}E(t)\mathrm{d}t \\
&+ C_3E(t_1) + \frac{(m_0-k)(p+2)}{k(p+1)}\int_{t_1}^{t_2}E(t)\mathrm{d}t \\
=& C_0E(t_1) + \left(3+2\epsilon+\frac{1}{2\epsilon}\right)\frac{(m_0-k)(p+2)}{k(p+1)}\int_{t_1}^{t_2}E(t)\mathrm{d}t,
\end{aligned}
\tag{3.32}
$$

where $C_0 = 2C_1 + 2C_p^2 + C_3$.

On the other hand, by (3.1) and (3.9), we have

$$
\begin{aligned}
\int_{t_1}^{t_2}2(p+1)\int_\Omega F(u,v)\mathrm{d}x\mathrm{d}t &\leq \int_{t_1}^{t_2}\frac{p+1}{p+2}\eta(k_1\|\nabla u\|_2^2 + k_2\|\nabla v\|_2^2)^{p+2}\mathrm{d}t \\
&\leq \int_{t_1}^{t_2}\frac{p+1}{p+2}\eta\left(\frac{2(p+2)}{p+1}E(t)\right)^{p+2}\mathrm{d}t \\
&\leq \int_{t_1}^{t_2}2\eta\left(\frac{2(p+2)}{p+1}E(0)\right)^{p+1}E(t)\mathrm{d}t,
\end{aligned}
$$

which implies

$$\begin{aligned}
&2\int_{t_1}^{t_2} E(t)\mathrm{d}t - 2(p+1)\int_{t_1}^{t_2}\int_\Omega F(u,v)\mathrm{d}x\mathrm{d}t\\
\geq&2\int_{t_1}^{t_2} E(t)\mathrm{d}t - 2\eta\left(\frac{2(p+2)}{p+1}E(0)\right)^{p+1}\int_{t_1}^{t_2} E(t)\mathrm{d}t\\
=&2\left[1-\eta\left(\frac{2(p+2)}{p+1}E(0)\right)^{p+1}\right]\int_{t_1}^{t_2} E(t)\mathrm{d}t.
\end{aligned} \tag{3.33}$$

Note that $E(0) < E_1$, we observe that

$$1-\eta\left(\frac{2(p+2)}{p+1}E(0)\right)^{p+1} > 0.$$

Thus, combining (3.32) with (3.33) yields

$$2\left[1-\eta\left(\frac{2(p+2)}{p+1}E(0)\right)^{p+1}\right]\int_{t_1}^{t_2} E(t)\mathrm{d}t \leq C_0E(t_1) + \left(3+2\epsilon+\frac{1}{2\epsilon}\right)\frac{(m_0-k)(p+2)}{k(p+1)}\int_{t_1}^{t_2} E(t)\mathrm{d}t. \tag{3.34}$$

Taking $\epsilon = \frac{1}{2}$ in (3.34), we have

$$\left[2-2\eta\left(\frac{2(p+2)}{p+1}E(0)\right)^{p+1} - \frac{5(m_0-k)(p+2)}{k(p+1)}\right]\int_{t_1}^{t_2} E(t)\mathrm{d}t \leq C_0E(t_1). \tag{3.35}$$

Denote

$$\lambda = 1-\eta\left(\frac{2(p+2)}{p+1}E(0)\right)^{p+1} - \frac{5(m_0-k)(p+2)}{2k(p+1)}. \tag{3.36}$$

We rewrite (3.35) as

$$2\lambda\int_t^\infty E(\tau)\mathrm{d}\tau \leq C_0E(t), \tag{3.37}$$

for every $t \in [0,\infty)$.

Since $\lambda > 0$ when $m_0 - k > 0$ is small enough, by Lemma 2.2, we obtain the following energy decay

$$E(t) \leq E(0)e^{1-2\lambda C_0^{-1}t}$$

for every $t \geq C_0(2\lambda)^{-1}$. The proof is completed. □

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